Advances in Smart Systems Research ISSN 2050-8662
Vol. 4. No. 1 : pp.17-24 : isrp15-007

Sangaku Presented by Augmented Reality: An Educational Tool for Geometry

Yoshiaki Fukumoto[1], Tensi Yanagimachi[2], Masatoshi Nozawa[3], Yoshiteru Ishida[4]

Department of Computer Science and Engineering,
Toyohashi University of Technology, Japan
{[1]fukumoto,[2]yanagimachi,[3]nozawa}@sys.cs.tut.ac.jp, [4]ishida@cs.tut.ac.jp

Abstract

Sangaku were media using wooden tablets to exchange knowledge of geometry in Edo period of Japan. We have been studying ICT to resurrect Sangaku as a new media to exchange knowledge of diagrams. This note reports how we used Augmented Reality (AR) to explore some beautiful theorems of geometry related to spheres of Sangaku. It is demonstrated that AR allows manual operations of removing or adding spheres, and continuous transformation of existing spheres, which has been impossible without diagram visualization by computers.

1. Introduction

Diagram is neither mere static pictures nor even movies. Diagram may be defined as operational geometric objects that allow a full use of human visual perception and human geometric reasoning. We have been studying extended graphs (networks) [9, 10], continuously transformable geometric objects [1, 6, 8, 10, 11], and demonstrated that they play an important role in knowledge elicitation, knowledge transfer and visual understanding. Since diagrammatic reasoning is the important human reasoning [13, 14], they even inspire a discovery of regularities and principles in geometry.

Diagrams have been playing an important role not only Engineering, but almost all the fields encompassing Chemistry (e.g., a structural formula), Physics (e.g., Feynman Diagrams), Mathematics (e.g., graphs, squares, moduli, commutative diagrams in category theory to mention only a few).

Among them, it is notable that *Wasan* (Japanese native mathematics) [w1] used geometric objects on *Sangaku* (wooden tablets explaining geometric findings) [2, 3]. Although it may be debatable whether the geometric objects used can be accommodated as diagrams, we have been studying them to be extended as diagrams using ICT.

We propose a new media of AR Sangaku [4, 5] resurrected by using Augmented Reality [12, w2, w3] with operational diagrams. Several computer aided operations are indeed possible from discrete operations of adding and removing spheres to continuous operations of transforming shape and size of existing spheres in the examples of geometric theorems found in SANGAKU.

2. Tangible Mathematics by extended *Sangaku*

Japanese culture may be biased to visual perception (in contrast to auditory) as exemplified by Japanese Manga, Animation, Video game, and significantly *Wasan* (Japanese traditional mathematics) as noted by *Sangaku* (wooden tablets with drawing of geometric theorem) [3, 4]. Also, it should be worth noting that Japanese culture places an importance on compactness, as known from electronic devices, stationary goods, and significantly Japanese *Haiku* and *Waka*. We can trace the roots of *Sangaku* in these two components: bias to visual perception and compact fashion.

Sangaku were dedicated to shrines and temples to appreciate the solution; to exchange geometric knowledge; or even to call for solutions. *Sangaku* had been playing a role of knowledge and intelligent media to exchange the geometric findings, and to appreciate beauty of geometric theorems. Geometry, a field of mathematics, requires diagrams extensively to reason, infer and prove theorems. Before computer age, diagrams must be drawn on papers, sand, leafs, or wooden tablet. Because they are two dimensional, geometric inference and reasoning based on the 2D diagrams must be limited. However, recent advent of ICT permits 3D realization, operations of geometric objects.

2.1. Augmented Reality (AR)

Augmented Reality (AR) or Extended Reality (ER) is a technology to augment the reality by adding images created by computers, thus allowing humans to perceive the augmented (or extended) reality. Virtual Reality (VR) places importance on a virtual world in which human can experience as if it were the real world. Mixed Reality (MR) is a mixture of virtual and (augmented) real world.

We use AR and ARToolKit [12]: its C/C++ programming library to support the development of the AR. AR prepares several markers with a specific pattern. A 3D object is displayed on the marker when the marker is recognized by a camera.

2.2. Sangaku Related to Spheres

We use geometric theorems of spheres in Sangaku as examples for demonstrating AR Sangaku system. Geometric theorems of spheres in Sangaku may be generalized as packing spheres in a large sphere where spheres inside the outer sphere touch with each other and the outer sphere where these spheres satisfy a fixed configuration. Hexlet Theorem [7, 15], for example, has two core spheres touching each other and are touching the outer sphere; while a chain of six spheres surrounding the trench of the two core spheres touch the two core spheres and each of them touches with two adjacent spheres forming a ring (Figure 1). Thirty spheres theorem [16] has 30 spheres of equal size touching an inner sphere of the larger size. Each of 30 spheres touches four adjacent spheres covering the inner sphere forming a polygon of dodecahedron (Figure 2). Twenty spheres theorem [2] has a configuration similar to the thirty sphere theorem.

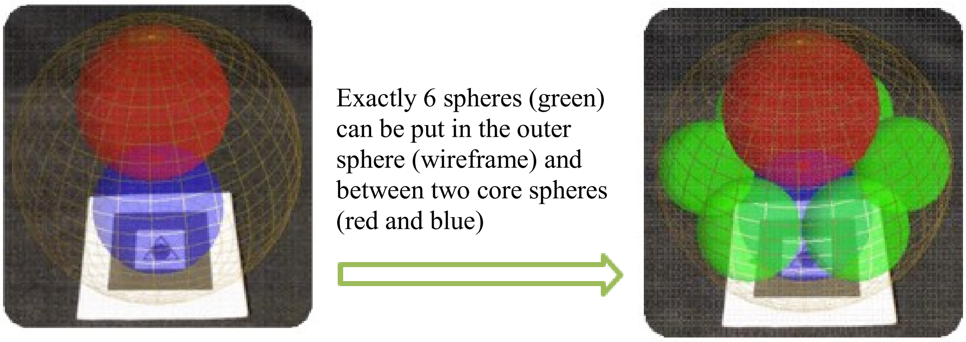

Figure 1. Hexlet theorem where outer sphere is shown with red wireframe; two core spheres colored red and blue (left); and six chain spheres green (right).

2.3. Six Chain Spheres (Hexlet Theorem)

Two core spheres touching each other (externally tangent) touch an outer sphere from inside (internally tangent). Six spheres with radius (r_1, r_2, …, r_6) surrounding and touching these two core spheres touch two adjacent chain spheres; and all the six chain spheres touch the outer sphere from inside satisfying the following formula:

$$\frac{1}{r_1}+\frac{1}{r_4}=\frac{1}{r_2}+\frac{1}{r_5}=\frac{1}{r_3}+\frac{1}{r_6} \quad (1)$$

where r_i (i=1..6) are radii of six chain spheres (Figure 1 *right*).
2.4. Thirty Spheres Touching a Center Sphere
Another theorem related to spheres can be found in the Sanpo Jojutu [16]. Thirty spheres of the same radius (r) are surrounding and touching a center sphere of the radius (R) satisfying the following formula:

$$\mathrm{R} = \sqrt{5}r \quad (2)$$

where each surrounding sphere touch adjacent four spheres (Figure 2).

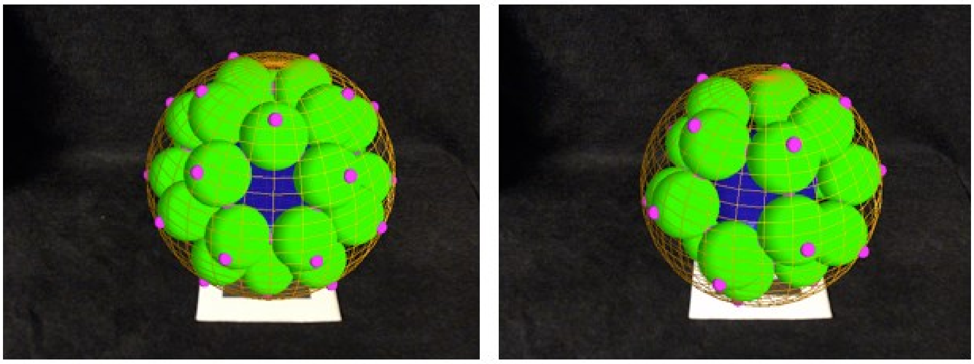

Figure 2. Thirty spheres (left) and twenty spheres (right) colored green touch each other and also the center sphere colored blue.

3. Operability with AR Sangaku

Operations for the AR *Sangaku* system can be done through four AR markers and keyboard input. The four AR markers are: a *wasan* marker, a rotation marker, a display marker and an inversion marker.

The *wasan* marker makes the system display 3D model of the diagram corresponding to the theory. Figure 3 shows overlaid images of the 3D model invoked by the *wasan* marker corresponding to the hexlet Theorem. Fig, 3 also shows the bar chart and the pentagon image invoked, respectively, by the display marker and the rotation marker. The rotation marker controls a posture the 3D model. It allows rotation on either *X*, *Y*, or *Z* axis. The display maker allows displaying quantitative information such as radii of spheres in an appropriate format.

The inversion maker displays an inverted image of the 3D model where the inversion is a mapping using a reference sphere (Figure 3). The inversion usually maps a sphere to a sphere, however, a sphere can be a plane (a sphere with the infinite radius) when the reference sphere is at a singular position.

Inversion in geometry is defined as a mapping from a point *P* of a geometric object to a point *P'* of the mapped object with respect to a reference circle with center *O* and the radius *R* so that the constraint:

$$OP \times OP' = R^2 \qquad (3)$$

is satisfied where the inverted point *P'* is on the half line *OP.*

Two kinds of operations are possible on the 3D model displayed: a continuous transformation and a discrete operation. Continuous transformation allows users to continuously change the size and the shape of spheres. In the hexlet theorem, the size of radii of two core spheres and six chain spheres can be continuously changed. The display maker allows finding regularities satisfied by the radii of spheres (It is already known that the sum of curvatures of the three pair spheres are equal as in the formula (1) [14, 16].).

On the other hand, the discrete operation allows users to remove an existing sphere and to add a new sphere on the 3D model. The inversion marker may be conveniently used to find a discrete regularity satisfied by the spheres (It is already known that only six chain spheres are possible and removing or adding a sphere resulted in violation of constraint of touching.). Why only 6 chain spheres exactly fit in may be diagrammatically recognized when all the spheres are inverted by a reference sphere whose center is the touching point of two core spheres: image before inversion (Figure 3 left) and after inversion (Figure 3 right). Touching relation among spheres is invariant, while two core spheres become two parallel planes (wireframe in Figure 3 right): and the outer sphere (orange wireframe in Figure 3 left) is placed inside these parallel planes (orange wireframe in Figure 3 right).

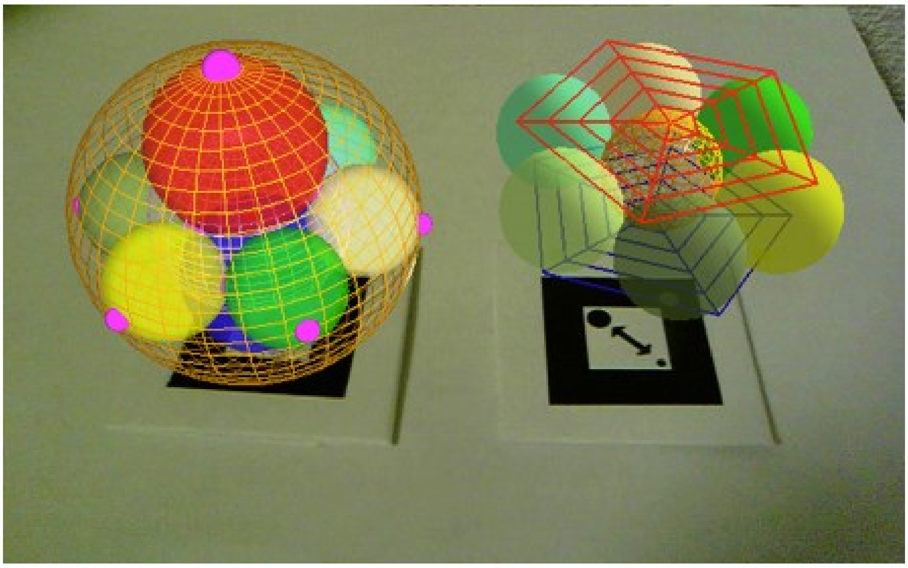

Figure 3. Hexlet theorem where outer sphere is shown with red wireframe; two core spheres colored red and blue; and six chain spheres light blue and green.

It is a significant feature of tangible mathematics to allow direct operations on the geometric objects. If we try to make one of chain sphere smaller and smaller on 3D model until it almost disappear, we can recognize that touching constraint by other spheres ceased to be satisfied. It can be also recognized by the discrete operation: if we remove one of six chain spheres and add two spheres in the space left; or shrink the left space to disappear, we can recognize satisfying the touching constraint would be impossible. The impossibility can be recognized by observing the inverted image that would change associating the above discrete operations.

4. Discussions

The objective of the AR Sangaku system is to assist studying geometry (3D geometry, in particular) in a operational, constructive and creative manner, as a method of E-learning geometry. This section discusses some possible examples of AR Sangaku other than an educational tool for 3D geometry.

As for the significance of the work for intelligent systems, we need to remind ourselves that design problems are of vital importance due to the difficulty in attaining creativity by machines.

One possible application is to use the AR Sangaku as a design tool of satellite orbit arrangement where geometric shapes such as circles and ellipses can be easily arranged relative to planets.

5. Conclusions

AR sangaku are new knowledge media to support *tangible mathematics* rooted at Japanese culture. AR sangaku is suited for 3D geometry, for geometric objects such as spheres inside a sphere is difficult to create as physical objects. With the support of ICT such as Augmented Reality, the geometric knowledge can be not only displayed as geometric objects but are allowed to be manipulated and even touched as if they were real existing objects; thus helps users to understand operationally and even encourage users to discover new knowledge and to create their own diagrams.

Acknowledgments

We would like to thank the reviewers for their valuable comments. Discussion on the importance and significance of the work in relation to intelligent systems has been added.

This presentation has been made possible by “Diagram Visualization DOJO” which is one of the collaborative projects between Toyohashi University of Technology and Colleges of Technology, which has been financially supported by Toyohashi University of Technology (TUT).

6. References

[1] M. Fujisawa and Y. Ishida, "Diagram, Dynamic Geometry and Sangaku," *Novel Algorithms and Techniques in Telecomm., Autom. and Ind. Electronics*, Springer, pp. 267-272, 2008.

[2] S. Fujita et al. eds., *Shinpeki Sanpo*, 1789 (in old Japanese).

[3] H. Fukagawa and T. Rothman, Sacred Mathematics: SANGAKU, Morikita Publishing, 2010 (in Japanese).

[4] Y. Fukumoto (2014) Operation Study on Sphere Geometry by AR, In poster presentation at SICE Chubu branch, Open Lab and Seminar for Young Researchers, at Toyota Commemorative Museum of Industry and Technology (TCMIT) in Nagoya, 15th Nov. 2014, in Japanese. (Students Award)

[5] Y. Fukumoto, Y. Ishida (2013) Exploring SANGAKU of Spheres by Augmented Reality In: International Workshop on Smart Info-Media Systems in Asia (SISA2013).

[6] X. S. Gao and Q. Lin, "MMP/Geometer," *Automated Deduction in Geometry*, pp. 44-66, 2002.

[7] S. Irisawa, Sangaku at Samukawa Shrine, 1822.

[8] Y. Ishida, "Diagrammatic reasoning by propagating constraints through geometric objects," *QR'95*, pp. 256-263, 1995.

[9] Y. Ishida, Computer WASAN JUKU: WASAN+ICT=Design, Lecture at Jishukan Senior High School as SS Technology and Science, 2012 (in Japanese), unpublished.

[10] Y. Ishida, Diagram, Communication and Education: Can we communicate without language? Invited Lecture at 3rd Hanoi Forum on ICT, 13th Oct. 2009, unpublished.

[11] N. Jakiw, *Geometer's Sketchpad, User Guide and Reference Manual*, Key Curriculum Press, Berkeley 1992.

[12] H. Kato, "Development of augmented reality building tool *ARToolKit*," *Technical Report of IEICE*, PRMU101(652), pp.79-86, 2002 (in Japanese).

[13] S. M. Kosslyn, *Image and Mind*, Harvard University Press, 1980.

[14] J. H. Larkin and H. A. Simon, "Why a diagram is (sometimes) worth a thousand words," *Cognitive Science*, vol. 11, 1987.

[15] F. Soddy, “The bowl of integers and the hexlet,” *Nature*, 139, pp. 77–79, 1937.

[16] Y. Yamamoto, *Sanpo Jojutu*, 1841 English by N. Nakamura: http://www.wasan.earth.lin

References on the web

[w1] WASAN http://www.wasan.org/ (accessed on 2015/1/9)

[w2] MapFan eye, http://www.mapfan.com/iphone/ar/ (accessed on 2015/1/9)

[w3] 3DSAR Games, http://www.nintendo.co.jp/3ds/hardware/ar/index.html (accessed on 2015/1/9)